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Note

Weak Tchebycheff Systems

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Let $(C[a, b], \|\cdot\|)$ be the space of all real-valued continuous functions on a compact interval $[a, b]$, endowed with the uniform norm $\|\cdot\|$. Let $\{u_i\}_{i=1}^n$ be linearly independent functions in $C[a, b]$. For each nonzero linear combination v of $\{u_i\}_{i=1}^n$, if v has at most $n-1$ zeros on $[a, b]$ (resp. v changes sign at most $n-1$ times), then $\{u_i\}_{i=1}^n$ is called a Tchebycheff system (resp. a weak Tchebycheff system). Furthermore if $\{u_i\}_{i=1}^n$ is a weak Tchebycheff system such that, for any nonzero linear combination v of $\{u_i\}_{i=1}^n$, $Z(v) = \{x \mid v(x) = 0\}$ is nowhere dense in $[a, b]$, we call it an integral Tchebycheff system (see [3]). As is well known, these systems play an important role as approximating functions in problems of best approximations. In this note, we are concerned with the following nonexistence theorem of best approximations by an infinite integral Tchebycheff system $\{u_i\}_{i=1}^\infty$, i.e., each system $\{u_i\}_{i=1}^k$, $k \in N$, is an integral Tchebycheff system.

THEOREM 1. *Let $\{u_i\}_{i=1}^\infty$ be an infinite integral Tchebycheff system on $[a, b]$ and let M be the closed linear subspace of $C[a, b]$ generated by $\{u_i\}_{i=1}^\infty$. Then the following are equivalent:*

(i) *M has the property that for each continuous function f outside M, there is no best approximation $\tilde{f}$ to f in M, i.e., $\|f-\tilde{f}\| = \inf_{g \in M} \|f-g\|$.*

(ii) *There is a positive integer k_0 such that each system $\{u_i\}_{i=1}^k$ is a Tchebycheff system for $k \geqq k_0$.*

To prove Theorem 1 we require some preliminary results.

THEOREM 2 (Cheney [2]). *Let $\{u_i\}_{i=1}^\infty$ be an infinite Markoff system on $[a, b]$, i.e., each system $\{u_i\}_{i=1}^k$, $k \in N$, is a Tchebycheff system. Let M be the closed linear subspace of $C[a, b]$ generated by $\{u_i\}_{i=1}^\infty$. Then M has the property* (i).

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We give a characterization of Tchebycheff systems in terms of integral Tchebycheff systems.

LEMMA. *Let G be the space spanned by an integral Tchebycheff system* $\{u_i\}_{i=1}^n$ *on* $[a, b]$. *Suppose that G contains a strictly positive function and contains two functions r, s such that*

$$\det\begin{pmatrix} r(a) & r(b) \\ s(a) & s(b) \end{pmatrix} \neq 0. \tag{T}$$

Then $\{u_i\}_{i=1}^n$ *is a Tchebycheff system.*

Proof. Suppose that $\{u_i\}_{i=1}^n$ is not a Tchebycheff system on $[a, b]$. By the condition (T), there is an $h \in G$ such that $h(a) = h(b) \neq 0$. By [4, Theorem 4–(4)] $f(a) = f(b)$ for any $f \in G$. This contradicts the condition (T).

Proof of Theorem 1. (ii) $\to$ (i) Since the proof in Theorem 2 can be used in the case of the infinite integral Tchebycheff system in Theorem 2, M has property (i).

(i) $\to$ (ii) First we will prove that some M_{k_1} contains a strictly positive function, where M_k denotes the space spanned by $\{u_i\}_{i=1}^k$. Suppose that, for each $n \in N$, any function of M_n has at least one zero on $[a, b]$. Since any function of M also has at least one zero on $[a, b]$, M does not contain 1. As is easily seen, 0, which belongs to M, is a best approximation to 1 by M. This contradicts the fact that M has the property (i).

Next we show that some M_{k_2} contains two functions r, s which satisfy the condition (T). Suppose to the contrary that no M_n contains functions that satisfy the condition (T). Since M contains a strictly positive function, M does not contain a continuous function $g(x)$ with $\|g\| = g(a) = -g(b)$ $(\neq 0)$ and 0 is a best approximation to g from M. This is contradictory to the property (i) of M. Hence M_k, $k \geqq \max\{k_1, k_2\}$, is a space spanned by an integral Tchebycheff system that contains a strictly positive function and contains two functions r, s satisfying

$$\det\begin{pmatrix} r(a) & r(b) \\ s(a) & s(b) \end{pmatrix} \neq 0.$$

By the lemma, each system $\{u_i\}_{i=1}^k$, $k \geqq \max\{k_1, k_2\}$, is a Tchebycheff system. This completes the proof.

Remark. If, in Theorem 1, "infinite integral Tchebycheff system" is replaced with "infinite weak Tchebycheff system," i.e., each system $\{u_i\}_{i=1}^k$, $k \in N$, is a weak Tchebycheff system, we can see that (i) does not always imply (ii) by the following example. Let us consider an infinite system $\{u_i\}_{i=0}^{\infty}$ in $C[0, 2]$ such that

$$u_0(x) = 1$$

$$u_{2i-1}(x) = \begin{cases} (x-1)^i & x \in [0, 1] \\ 0 & x \in [1, 2], \quad i = 1, 2, ... \end{cases}$$

$$u_{2i}(x) = \begin{cases} 1 & x \in [0, 1] \\ x^{i^2} & x \in [1, 2], \quad i = 1, 2, ... \end{cases}$$

By [1, Theorem 4], $\{u_i\}_{i=0}^{\infty}$ is a weak Tchebycheff system. Since $Z(u_{2i-1}) = [1, 2]$, $i = 1, 2, ...$, $\{u_i\}_{i=0}^{\infty}$ is not an infinite integral Tchebycheff system. By Muntz's Theorem (see [2, p. 197]), $\{1\} \cup \{(x-1)^i\}$ is fundamental in $C[0, 1]$ and $\{1\} \cup \{x^{i^2}\}$ is not fundamental in $C[1, 2]$. Since each $u_{2i-1}(x)$, $i = 1, 2, ...$, vanishes on $[1, 2]$, the closed subspace M generated by $\{u_i\}_{i=0}^{\infty}$ of $C[0, 2]$ consists of all real-valued continuous functions whose restrictions to $[1, 2]$ belong to the closed subspace M_1 generated by $\{1\} \cup \{x^{i^2}\}$. From this fact, for any continuous function f outside M, we have

$$\inf_{g \in M} \sup_{x \in [0,2]} |f(x) - g(x)| = \inf_{h \in M_1} \sup_{x \in [1,2]} |f(x) - h(x)|.$$

Since, in $C[1, 2]$, f does not have a best approximiation $\tilde{f}$ from M_1 by Theorem 2, there does not exist a best approximation to f from M in $C[0, 2]$. Hence M has property (i).

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